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Tensor Products of Banach Spaces and Weak Type (p, p) Multipliers

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DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

In 1950 G. G. Lorentz initiated the study of certain classes of function spaces associated with a measure space (X, Σ, μ) (cf. [12, 14, 15]). Particular examples—the so-called Lorentz L^{pq}-spaces—now play important roles in both harmonic analysis and abstract interpolation space theory. This volume of papers in celebration of Lorentz' sixty-fifty birthday, therefore, would seem to be an appropriate place for yet another series of results confirming the role of the L^{pq}-spaces. For all unexplained notation and terminology see [2, 9, or 10].

Let (X, Σ, μ) be a totally σ-finite measure space. If G is a locally compact group and μ a fixed left invariant Haar measure, G will be so restricted also. In one version of the Lorentz L^{pq}-spaces one defines $L^{pq}(X, \Sigma, \mu)$ as the Banach space of (equivalence classes of) μ-measurable functions f on X for which

$$\|f\|_{pq} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \, (dt/t)\right)^{1/q}, & 1 < p < \infty, \ 1 \leq q < \infty, \\ \sup_{0<t<\infty} (t^{1/p} f^{**}(t)), & 1 < p \leq \infty, \ q = \infty, \end{cases} \tag{1}$$

is finite. Up to equivalence of norms, i.e., with constants depending only on p,

$$L^{pp}(X, \Sigma, \mu) = L^p(X, \Sigma, \mu).$$

One crucial property of these spaces is that, up to equivalence of norms, a linear operator $T: L^p(Y, \Omega, \nu) \to L^{p\infty}(X, \Sigma, \mu)$ is bounded if and only if T is of Weak Type (p, p). In contrast, by definition $T: L^p(Y, \Omega, \nu) \to L^p(X, \Sigma, \mu)$

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is bounded if and only if T is of Strong Type (p, p). The terminology stems part from the fact that

$$L^{pr}(X, \Sigma, \mu) \subseteq L^{ps}(X, \Sigma, \mu), \qquad 1 \leqslant r \leqslant s \leqslant \infty. \tag{2}$$

When G is a locally compact group, the strong type (p, p) operators $T: L^p(G) \to L^p(G)$ satisfying $T(f * k) = (Tf) * k$ for all f in $L^p(G)$ and k in $\mathscr{K}(G)$ will be denoted by $Cv^p(G)$, and the corresponding operators $T: L^p(G) \to L^{p\infty}(G)$, $1 < p < \infty$, by $Cv_\omega{}^p(G)$. The closure of $L^1(G)$ in $Cv^p(G)$ (resp. $Cv_\omega{}^p(G)$) will be denoted by $cv^p(G)$ (resp. $cv_\omega{}^p(G)$). We set

$$A^p(G) = P(L^{p'}(G) \otimes_\gamma L^p(G)), \qquad 1 \leqslant p \leqslant \infty, \tag{3}$$

$$A_\omega{}^p(G) = P(L^{p'1}(G) \otimes_\gamma L^p(G)), \qquad 1 < p < \infty, \tag{4}$$

where $1/p + 1/p' = 1$ and, in (3), $L^\infty(G)$ is to be replaced by $\mathscr{C}_0(G)$ (cf. [6]). In view of (2),

$$A_\omega{}^p(G) \subseteq A^p(G) \subseteq \mathscr{C}_0(G). \tag{5}$$

In a series of papers [8], [9], [10] we have developed a new approach to L^p-convolution operator theory starting with so-called Varopoulos spaces $V^\alpha(X, Y) = \mathscr{C}_0(X) \otimes_\alpha \mathscr{C}_0(Y)$ and deriving from them, for every locally compact group G, Banach spaces $\mathscr{V}^\alpha(G)$ satisfying

$$A(G) \subseteq \mathscr{V}^\alpha(G) \subseteq \mathscr{C}_0(G). \tag{6}$$

In the very important special case $\alpha = \alpha_{p'q'}$, $1 \leqslant p \leqslant q \leqslant \infty$, the results of [9] and [10] show that the corresponding spaces $\mathscr{V}^{pq}(G)$ satisfy

(a) $\mathscr{V}^{pp}(G) = A^p(G)$, $1 \leqslant p \leqslant \infty$;

(b) $\mathscr{V}^{pq}(G) = A(G) = A^{22}(G)$, $1 \leqslant p \leqslant 2$, $2 \leqslant q \leqslant \infty$;

(c) $\mathscr{V}^{pq}(G) = \mathscr{V}^{1q}(G)$, $1 \leqslant p < q < 2$;

(d) $f \to \check{f}$ is an isomorphism from $\mathscr{V}^{pq}(G)$ onto $\mathscr{V}^{q'p'}(G)$.

Parts (c) and (d) hold for all G, but (a) and (b) are known only for G amenable. What is interesting is that (a), (b), and (c) are consequences of deep results from classical Banach space theory. If, following Doss ([5]), we use now a deep result of Stein from harmonic analysis ([18]), we can complete the identification of $\mathscr{V}^{pq}(G)$, at least when G is abelian. The main result of this paper is the following

MAIN THEOREM. *Let G be a locally compact abelian group. Then*

$$\text{(i)}\quad \mathscr{A}^{rp}(G) = A_\omega^{p}(G),$$
$$\text{(ii)}\quad \mathscr{L}^{pr}(G) = cv_\omega^{p}(G), \tag{7}$$
$$\text{(iii)}\quad (\mathscr{A}^{rp}(G))^* = Cv_\omega^{p}(G),$$

provided $1 \leqslant r < p < 2$. In particular, $A_\omega^p(G)$ is a Banach algebra under pointwise multiplication.

The equalities in (i), (ii), and (iii) hold up to equivalence of norms with constants depending possibly on the group G as well as p. The theorem provides yet another solution to the problem 9.1 posed by Eymard in [6].

The term multiplier is used in the title of this paper whereas convolution operator is implied in the notation $Cv^p(G)$, $Cv_\omega^p(G)$. This double terminology reflects the two ways of thinking of the operators in $Cv^p(G)$ and $Cv_\omega^p(G)$. When G is abelian denote by Γ its character group, and by Γ_d the group Γ equipped with the discrete topology, so that then Γ_d is the character group of the Bohr compactification bG of G. The Fourier Transform will always be denoted by $\mathscr{F}$. An operator T from $L^p(G)$ into $L^p(G)$ or into $L^{p\infty}(G)$ satisfies $T(f * k) = (Tf) * k$ for all f in $L^p(G)$ and k in $\mathscr{K}(G)$ if and only if there exists ϕ in $L^\infty(\Gamma)$ such that $\mathscr{F}(Tf) = \phi \cdot \mathscr{F}(f)$ for all f in $L^p(G)$ (slight modifications needed if $p > 2$). With obvious notation we write $M^p(\Gamma)$ and $M_\omega^p(\Gamma)$ for the set of all such ϕ and put

$$\|\phi\|_{M^p(\Gamma)} = \|T\|_{Cv^p(G)}, \quad \|\phi\|_{M_\omega^p(G)} = \|T\|_{Cv_\omega^p(G)}. \tag{8}$$

There are analogous definitions replacing Γ by Γ_d.

COROLLARY. *"Bochner–Eberlein." Let ϕ be a continuous function on Γ. Then ϕ belongs to $M_\omega^p(\Gamma)$ if and only if ϕ belongs to $M_\omega^p(\Gamma_d)$. Thus*

$$M_\omega^p(\Gamma) \cap \mathscr{C}(\Gamma) = M_\omega^p(\Gamma_d) \cap \mathscr{C}(\Gamma)$$

up to equivalence of norms with constants depending possibly on Γ and p.

The proof of the Main Theorem and its corollary proceeds in several stages. The basic idea is to show that $\mathscr{L}^{pr}(G) = cv_\omega^p(G)$, $1 \leqslant r < p < 2$, up to equivalence of norms, by passing to bG where it is known that $(\mathscr{A}^{rp}(bG))^* = Cv_\omega^p(bG)$ up to equivalence of norms. The main theorem then follows easily (cf. Section 4).

It will be convenient to collect together here some known and some possibly less well-known properties of the L^{pq}-spaces. The dual spaces $(L^{pq})^*$ behave much like the Lebesgue L^p-spaces:

$$(L^{pq}(X, \Sigma, \mu))^* = L^{p'q'}(X, \Sigma, \mu), \qquad \begin{array}{l} 1 < p < \infty, \\ 1 \leqslant q < \infty. \end{array} \tag{9}$$

In terms of interpolation space theory:

$$(L^1, L^\infty)_{\theta,q;K} = L^{pq}, \qquad \theta = 1 - 1/p, \tag{10}$$

isometrically; more generally

$$(L^{p_0q_0}, L^{p_1q_1})_{\theta,q;K} = (L^{p_0q_0}, L^{p_1q_1})_{\theta,q;J} = L^{pq}, \quad 1/p = 1 - \theta/P_0 + \theta/p_1, \tag{11}$$

up to equivalence of norms when $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ (cf. [2] Section 3.3). Let $\mathfrak{A}$ be a subspace of $\bigcap_{1 \leqslant p < \infty} L^p(X, \Sigma, \mu)$ dense in each space $L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. When q, r, s, and θ satisfy

$$1/q = (1 - \theta)/r + \theta/s, \qquad 0 < \theta < 1, \qquad 1 \leqslant q, r, s \leqslant \infty, \tag{12}$$

define $\|(\cdot)\|_{\theta,r,s}$ on $\mathfrak{A}$ by

$$\| \phi \|_{\theta;r,s} = \inf \sum_n (\| \phi_n \|_{L^r})^{1-\theta} (\| \phi_n \|_{L^s})^\theta, \qquad \phi \in \mathfrak{A}, \tag{13}$$

the infimum being taken over all finite series representations $\phi = \sum_n \phi_n$ with $\{\phi_n\} \subseteq \mathfrak{A}$. Then we can show

PROPOSITION 1. *When q, r, s, and θ satisfy* (12), $\|(\cdot)\|_{L^{q1}}$ *and* $\|(\cdot)\|_{\theta;r,s}$ *define equivalent norms on $\mathfrak{A}$, the constants depending only on q, r, s, and θ.*

Proof. Essentially this follows from (11) together with the fact that the Bishop interpolation space construction ([1, p. 470]) yields the maximal scale of spaces ([13, p. 109]) which is known to coincide, up to equivalence of norms, with the Peetre family $(\cdot, \cdot)_{\theta,1;1}$.

1

Throughout the remainder of this paper G will denote a locally compact abelian group. The proof of the following theorem uses only simple modifications of the proof of the corresponding results for $A^p(G)$ and $Cv^p(G)$ (cf. [6, 7]).

THEOREM 1. *Let G be a locally compact abelian group and $1 < p < \infty$. Then*

(i) *for each T in $Cv_\omega{}^p(G)$ there is a net $\{\phi_\alpha\}$ in $L^1(G)$ such that $\|\phi_\alpha\| \leq \|T\|$ in $Cv_\omega{}^p(G)$ while $\phi_\alpha \to T$ in the strong operator topology;*

(ii) *$(A_\omega{}^p(G))^* = Cv_\omega{}^p(G)$ isometrically setting*

$$\langle T, f\rangle = \sum_n \langle f_n, Tg_n\rangle \tag{14}$$

for T in $Cv_\omega{}^p(G)$ and $f = P(\sum_n f_n \otimes g_n)$ in $A_\omega{}^p(G)$.

Proof. (i) A careful reading of the proof given on page 2 in [7] for the case when T maps $L^p(G)$ into $L^q(G)$ shows that the following properties of $L^q(G)$ are used:

(a) $q \neq 1$ so that then $L^q(G) = (L^{q'}(G))^*$,

(b) $L^{q'}(G)$ is a Banach $L^\infty(G)$-module under multiplication.

But, by (9), $L^{p\infty}(G)$ is a dual space if $1 < p \leq \infty$. On the other hand, by interpolation and (10), $L^{p'1}(G)$ is a Banach $L^\infty(G)$-module under pointwise multiplication since $L^1(G)$ and $L^\infty(G)$ are. With these observations the proof on page 2 of [7] for $T: L^p(G) \to L^q(G)$ can be carried over to $T: L^p(G) \to L^{p\infty}(G)$.

(ii) Although we have followed Eymard in our definition of $A_\omega{}^p(G)$ and in (14), the proof in [7, p. 4] carries over to $A_\omega{}^p(G)$ and $Cv_\omega{}^p(G)$ because once again properties (a) and (b) are the vital ones.

2

If $\sigma_b: G \to bG$ is the canonical injection, then $\pi_b: \phi \to \phi \circ \sigma_b$ defines an isometric mapping of $\mathscr{V}^{pq}(bG)$ in $\mathscr{W}^{pq}(G)$, $1 \leq p \leq q \leq \infty$, (cf. [10, Theorem (3.13)]. Identifying $\mathscr{L}^{qp}(G)$ with a closed subspace of $(\mathscr{W}^{pq}(G))^*$ $(= \mathscr{L}^{qp}(G))^{**})$ we obtain a norm-decreasing mapping

$$\pi_b{}^*: \mathscr{L}^{qp}(G) \to (\mathscr{V}^{pq}(bG))^*, \tag{15}$$

[10, Corollary (2.13)]. Since the Fourier Transform is $1-1$ on $\mathscr{L}^{qp}(G)$, $\pi_b{}^*$ clearly is $1-1$ in (15); in fact,

$$\mathscr{F}(\pi_b{}^*(f)) = \mathscr{F}(f)|_{\Gamma_d}, \qquad f \in \mathscr{L}^{qp}(G).$$

THEOREM 2. *For $1 \leq p \leq q \leq \infty$ the mapping $\pi_b{}^*$ is an isometry from $\mathscr{L}^{qp}(G)$ into $(\mathscr{V}^{qp}(G))^*$.*

Proof. Since $\mathscr{W}^{pq}(G) = (\mathscr{L}^{qp}(G))^*$, it is enough to show that the unit ball of $\mathscr{V}^{pq}(bG)$ is dense in the unit ball of $\mathscr{L}^{pq}(G)$ in the $\sigma(\mathscr{W}^{pq}(G), \mathscr{L}^{qp}(G))$-topology. But, by the results of Section 3 in [11], this property is true of all tensorial norms α including the tensorial norm $\alpha_{p'q'}$ used to define $\mathscr{V}^{pq}(bG)$ and $\mathscr{W}^{pq}(G)$.

3

In this section we shall establish the analog of Theorem 2 for the spaces $cv_\omega{}^p(G)$ and $Cv_\omega{}^p(bG)$. Although the proof given owes much to the work of Coifman–Weiss [3], de Leeuw [4], and Saeki [17], it does offer a slightly different approach.

To each f in $L^1(G)$ there corresponds an operator K_f on $\mathscr{C}(bG)$ defined by

$$(K_f\phi)(\xi) = \int_G f(x)\, \phi(\sigma_b(x^{-1})\xi)\, dx, \qquad \phi \in \mathscr{C}(bG), \qquad \xi \in bG. \tag{16}$$

Clearly the mapping $\kappa: f \to K_f$ is $1 - 1$ and

$$\mathscr{F}(K_f) = \mathscr{F}(f)|_{\Gamma_d},$$

i.e., κ coincides with $\pi_b{}^*$ on $L^1(G)$.

THEOREM 3. *For $1 < p < \infty$ the mapping κ extends to a bounded linear mapping κ: $cv_\omega{}^p(G) \to Cv_\omega{}^p(bG)$ such that*

$$A_p \|f\|_{cv_\omega^p(G)} \leqslant \|\kappa f\|_{Cv_\omega^p(bG)} \leqslant B_p \|f\|_{cv_\omega^p(G)}, \qquad f \in L^1(G), \tag{17}$$

with A_p and B_p constants depending only on p.

It is enough to establish (17) for a function f in $\mathscr{K}(G)$ with compact support, say, C. Let H be a compactly generated subgroup of G containing C, and let σ_H: $H \to G$ be the canonical injection. The proof of (17) will be broken down into several steps. Notice first that, whenever H is any closed subgroup of G, there is an injection π_H: $L^1(H) \to M(G)$ $(= (\mathscr{C}_0(G))^*)$ given by

$$\int_G \phi(x)\, d(\pi_H f) = \int_H (\phi \circ \sigma_H)(\xi) f(\xi)\, d\xi. \tag{18}$$

PROPOSITION 2. *For any closed subgroup H of G the mapping π_H extends to a bounded linear mapping $\pi_H : cv_\omega{}^p(H) \to Cv_\omega{}^p(G)$ such that*

$$A_p' \|f\|_{cv_\omega^p(H)} \leqslant \|\pi_H f\|_{Cv_\omega^p(G)} \leqslant B_p' \|f\|_{cv_\omega^p(H)}, \qquad f \in L^1(G), \tag{19}$$

with A_p' and B_p' depending only on p.

Proof. To establish to right-hand inequality it is enough to estimate

$$\sup\{|\langle\psi, (\pi_H f) * \phi\rangle|: \phi, \psi \in \mathscr{K}(G), \|\phi\|_{L^p(G)}, \|\psi\|_{L^q(G)} \leqslant 1\}, \qquad f \in \mathscr{K}(H),$$

where $1/p + 1/q = 1$ (cf. (9)). Now, by (18) and invariance of Haar measure,

$$\begin{aligned}\langle\psi, (\pi_H f) * \phi\rangle &= \int_G \psi(x) \left[\int_H \phi(\xi^{-1}x) f(\xi)\, d\xi\right] dx \\ &= \int_H s(\eta) \left\{\int_G \psi(x\eta) \left[\int_H \phi(\xi^{-1}x\eta) f(\xi)\, d\xi\right] dx\right\} d\eta \\ &= \int_G \left\{\int_H \left[\psi(x\eta) \int_H \phi(\xi^{-1}x\eta) f(\xi)\, d\xi\right] s(\eta)\, d\eta\right\} dx,\end{aligned}$$

for any s in $L^1(H)$ such that $\int_H s = 1$. But, since H has the property P_1, to each $\epsilon > 0$ there corresponds s in $L^1(H)$ so that

$$\begin{aligned}&\left| \int_G \left\{\int_H \left[\psi(x\eta) \int_H \phi(\xi^{-1}x\eta) f(\xi)\, d\xi\right] s(\eta)\, d\eta\right\} dx \right| \\ &\qquad \leqslant (1 + \epsilon) \left(\int_{G/H} \left| \int_H \psi(x\eta) \left[\int_H \phi(\xi^{-1}x\eta) f(\xi)\, d\xi\right] d\eta \right| d\dot{x}\right),\end{aligned}$$

Haar measure on G, H and G/H suitably adjusted (cf. [16, pp. 115, 168]). Hence

$$|\langle\psi, (\pi_H f) * \phi\rangle| \leqslant (1 + \epsilon) \|f\|_{cv_\omega^p} \left(\int_{G/H} \|\psi_x\|_{L^q(H)} \|\phi_x\|_{L^p(H)} d\dot{x}\right),$$

where $\phi_x(\eta) = \phi(x\eta)$. Since

$$\int_{G/H} (\|\phi_x\|_{L^p(H)})^p\, d\dot{x} = \int_G |\phi(x)|^p\, dx,$$

it follows easily from Holder's inequality and Proposition 1 that

$$|\langle\psi, (\pi_H f) * \phi\rangle| \leqslant (1 + \epsilon) A_p' \|f\|_{cv_\omega^p} [\|\psi\|_{L^q(G)} \|\phi\|_{L^p(G)}],$$

establishing the right-hand inequality in (19).

For the left-hand inequality we have to estimate

$$\sup\{|\langle\psi, f * \phi\rangle|: \phi, \psi \in \mathscr{K}(H), \|\phi\|_{L^p(H)}, \|\psi\|_{L^q(H)} \leqslant 1\}, \qquad f \in \mathscr{K}(H).$$

Choose Φ, $\Psi \in \mathscr{K}(G)$ so that $\Phi|_H = \phi$, $\Psi|_H = \psi$. For each compact neighborhood U of e in G/H set

$$\sigma_u = (1/m(U))^{1/q} \chi_u, \qquad \tau_u = (1/m(U))^{1/p} \chi_u,$$

where χ_u is the characteristic function of U and $m(U)$ its Haar measure (in G/H). Then

$$|\langle\sigma_u\Psi, (\pi_H f) * (\tau_u\Phi)\rangle| \leqslant \| \pi_H f \| \, \| \sigma_u\Psi \|_{L^{q1}(G)} \| \tau_u\Phi \|_{L^p(G)},$$

while

$$\lim_U \langle\sigma_u\Psi, (\pi_H f) * (\tau_u\Phi)\rangle$$
$$= \lim_U \frac{1}{m(U)} \int_{G/H} \chi_u(x) \left[\int_H \Psi(x\eta)(\pi_H f * \Phi)(x\eta)\, d\eta\right] dx$$
$$= \int_H \psi(\eta)(f * \phi)\, d\eta = \langle\psi, f * \phi\rangle.$$

Since also

$$\lim_U \| \tau_u\Phi \|_{L^p(G)} = \| \phi \|_{L^p(H)},$$

an easy application of Proposition 1 shows that

$$\lim_U \sup \| \sigma_u\Psi \|_{L^{q1}(G)} \leqslant 1/B_p' \| \psi \|_{L^{q1}(H)},$$

for some constant B_p'. The left-hand inequality now follows.

Entirely analogously there is a bounded linear mapping

$$\pi_{bH}\colon cv_\omega{}^p(bH) \to Cv_\omega{}^p(bG)$$

and a pair of inequalities corresponding to (19). Theorem 3 now follows immediately from this last pair of inequalities, (19), and the next result.

PROPOSITION 3. *Let H be a compactly generated subgroup of G. Then, for $1 < p < \infty$, the mapping κ extends to an isometry*

$$\kappa\colon cv_\omega{}^p(H) \to Cv_\omega{}^p(bH),$$

from $cv_\omega{}^p(H)$ into $Cv_\omega{}^p(bH)$.

The corresponding result for $cv^p(H)$ is due (essentially) to de Leeuw [4] since any compactly generated locally compact abelian group is isomorphic to a group of the form $\mathbf{R}^n \times \mathbf{Z}^m \times \Phi$ with Φ compact. de Leeuw's proof carries over easily to the setting of Proposition 3. We omit the details.

4

The proof of the Main Theorem is now almost trivial. For, by Theorems 2 and 3, there exist mappings

$$\pi_b{}^*\colon \mathscr{L}^{pr}(G) \to (\mathscr{V}^{rp}(bG))^*, \qquad \kappa\colon cv_\omega{}^p(G) \to Cv_\omega{}^p(bG),$$

the first an isometry and the second a norm equivalence mapping (cf. (17)), such that $\pi_b{}^*$ coincides with κ on the subspace $L^1(G)$ dense in both spaces. But, by Corollary 6.2 in [9],

$$(\mathscr{V}^{rp}(bG))^* = Cv^{pr}(bG),$$

isometrically; on the other hand, using a result of Stein [18], Doss has shown that

$$Cv^{pr}(bG) = Cv_\omega{}^p(bG), \qquad 1 \leqslant r < p < 2,$$

[5]. Thus $\mathscr{L}^{pr}(G) = cv_\omega{}^p(G)$ up to norm equivalence, establishing part (ii) of the Main Theorem.

Now by part (i) of Theorem 1, $A_\omega{}^p(G)$ embeds isometrically in $(cv_\omega{}^p(G))^*$; and $\mathscr{V}^\alpha(G)$ always is a closed subspace of $(\mathscr{L}^{\alpha'}(G))^*$ $(= \mathscr{W}^\alpha(G))$ (cf. [10, Corollary (2.13)]. Hence $A_\omega{}^p(G) = \mathscr{V}^{rp}(G)$ and $(\mathscr{V}^{rp}(G))^* = Cv_\omega{}^p(G)$ completing the proof of the Main Theorem.

With the identifications in the Main Theorem, the corollary is just one special case of the results in [11].

Remark. Theorem 3, a critical step in the proof of the Main Theorem, is a special case of the corollary of the Main Theorem. It seems likely (better still, it is to be desired) that this step and the use of structure theory for G be avoided by finding a version of the Stein theorem in the context of the spaces $\mathscr{C}_0(X) \otimes_\alpha \mathscr{C}_0(Y)$, $L^1(G_1) \otimes_{\alpha'} L^1(G_2)$.

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